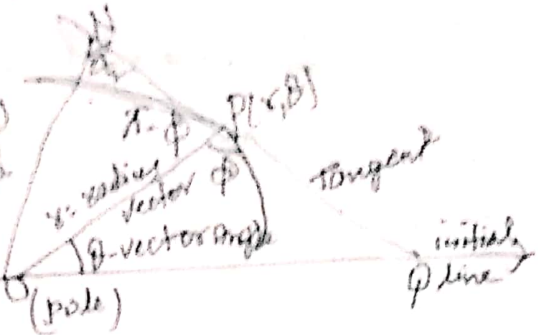
B.Sc-Part-II (H), Paper-III Diff. Calculus, Tangent and Normals 16-07-2020

Q. 1. Prove that (i) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

(ii) $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$



Soln: Let P be a given point (r, θ) on the curve $r = f(\theta)$

Let ON be the perpendicular from the pole O on the tangent at P. Let $ON = p$.

Then, from the right angled $\triangle OPN$

$$ON = OP\sin(\psi - \theta) \Rightarrow p = r\sin\phi \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$\Rightarrow ON = \frac{1}{r^2}(1 + \cot^2\phi)$$

$$= \frac{1}{r^2}\left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}, \text{ as } \tan\phi = r\frac{d\theta}{dr}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

If $u = \frac{1}{r^2}$ then, $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, then $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$.

Proved

Q.2. If the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at any point on it cuts the axes OX and OY at P and Q respectively, prove that

$$OP + OQ = \text{constant}.$$

Soln. It is given that the eqn of the curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \quad \text{—— (1)}$$

Diff. with respect to x, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

The eqn of the tangent to (1) at (x, y) is

$$Y - y = -\frac{\sqrt{y}}{\sqrt{x}}(X - x)$$

$$\Rightarrow \frac{Y}{\sqrt{y}} - \sqrt{y} = -\frac{X}{\sqrt{x}} + \sqrt{x}$$

$$\Rightarrow \frac{X}{\sqrt{x}} + \frac{Y}{\sqrt{y}} = \sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$\Rightarrow \frac{X}{\sqrt{ax}} + \frac{Y}{\sqrt{ay}} = 1$$

$\therefore OP = \sqrt{ax}$ and $OQ = \sqrt{ay}$

Hence, $OP + OQ = \sqrt{a}(\sqrt{x}+\sqrt{y}) = \sqrt{a}\cdot\sqrt{a}$ [by ①]

$\underset{=a}{}$

$\Rightarrow \sqrt{x}+\sqrt{y} = \sqrt{a}$ which is constant.

$\Rightarrow OP + OQ = a$, which is constant.

Q. 3. Prove that the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{a}\right)^n = 2$ touches the straight line

$\frac{x}{a} + \frac{y}{b} = 2$ at the point (a, b), whatever be the value of n.

Soln: It is given that the eqn of the curve is

$$\frac{x^n}{a^n} + \frac{y^n}{b^n} = 2$$

Differentiating with respect to x, we get

$$n\frac{x^{n-1}}{a^n} + n\frac{y^{n-1}}{b^n}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^n}{a^n} \times \frac{x^{n-1}}{y^{n-1}}$$

At the point (a,b) of the curve, $\frac{dy}{dx} = -\frac{b^n}{a^n}\frac{a^{n-1}}{b^{n-1}} = -\frac{b}{a}$

Hence, the eqn of the tangent to the curve at point (a,b) is

$$y - b = \left(\frac{dy}{dx}\right)_{a,b}(x-a) = -\frac{b}{a}(x-a)$$

$$\Rightarrow \frac{y}{b} - 1 = -\frac{x}{a} + 1 \Rightarrow \frac{x}{a} + \frac{y}{b} = 2$$

Which is independent of n.

Proved

16-7-2020